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Two scale factor universality in the spherical model

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Abstract. A universal relation, proposed by Stauffer, Ferer and Wortis, between the scale factors in the scaling functions for the free energy and two-spin correlation function, is shown to be valid for the spherical model with general pair interactions, provided an appropriate definition of the correlation length is adopted. The hypothesis remains valid for the spherical model and numerically for the three-dimensional Ising model, when extended to include the scale factor for the magnetic field.

1. Introduction

Work of the last few years makes it likely that thermodynamic properties and long-range correlations of a system in the neighbourhood of a critical point can be described in terms of universal scaling functions†. The details of the particular interactions of different systems within one universality class are then to be taken into account by choosing appropriate scales for the variables in these functions. Recently, Stauffer, Ferer and Wortis (1972, Ferer *et al* 1973a, b) proposed, as a hypothesis, a universal relation between the scale factors appearing in the scaling functions for the zero field ($h = 0$) free energy and two-point correlation function. With this relation the number of independent scale factors needed for zero field is reduced from three to two. Ferer *et al* based their hypothesis on the assertion, that the leading singular part of the free energy associated with a volume ξ^d, where ξ is the correlation length, should be a universal quantity, independent of the irrelevant details of the Hamiltonian. The hypothesis was checked against exact results for the two-dimensional Ising model and against numerical data derived from series for the three-dimensional Ising and Heisenberg models, all on Bravais lattices. Various experimental data were also examined (see also Stauffer 1973). We may note incidentally, that the hypothesis also holds for the Ising model on a honeycomb lattice, a case of a non-Bravais lattice (Houtappel 1950, Fisher and Burford 1967). Aharony (1974) has considered the problem in the framework of the ϵ expansion technique of the renormalization group approach (Wilson and Kogut 1974, Wilson and Fisher 1972), where $\epsilon = 4-d$ and d is the dimensionality. He verified the hypothesis to order ϵ^2 for isotropic n-component spin vector systems with short-range interactions.

However, the limit $n \to \infty$ can be treated exactly for all dimensions since it corresponds to the spherical model (Berlin and Kac 1952, Stanley 1968, Kac and Thompson 1971). To state our present result we denote the density of spins by ρ_s, the leading singular

† The universality hypothesis for the scaling functions (Watson 1969, Kadanoff 1971) is strongly supported by the results of Wilson's renormalization group approach (Brezin *et al* 1973, Fisher and Aharony 1973).

part of the free energy per spin, measured in units of $k_B T$, by F_s and the correlation length by ξ. Then we will prove that the quantity

$$Z = \rho_s F_s \xi^d \tag{1}$$

indeed becomes universal as the temperature T and the reduced magnetic field h approach their critical values T_c and zero, respectively. In fact Z depends only on the particular locus on which the critical point is approached. This confirms and also extends the basic statement of Stauffer *et al* (which was restricted to the locus $h = 0$). However, to establish a precise result the correlation length ξ must be defined in terms of the second moment of the two-point correlation function in the case of short-range interactions, but in terms of appropriate lower-order moments where there are long-range ferromagnetic interactions decaying as $r^{-d-\sigma}$.

2. Spherical model free energy

2.1. General formulation

We consider a spherical model on a Bravais lattice with an interaction energy $J(\boldsymbol{l}-\boldsymbol{j})S_{\boldsymbol{l}}\,.\,S_{\boldsymbol{j}}$ between the spins $S_{\boldsymbol{l}}$ and $S_{\boldsymbol{j}}$ at positions $\boldsymbol{l}$ and $\boldsymbol{j}$, with $J(\boldsymbol{l}) = J(-\boldsymbol{l})$. As usual we define the Fourier transform

$$\hat{J}(\boldsymbol{k}) = \sum_{\boldsymbol{l}} J(\boldsymbol{l}) \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{l}). \tag{2}$$

Then up to analytic terms in the temperature the free energy can be written (Joyce 1972)

$$F_s(x, h) = \frac{1}{2}\frac{V_a}{(2\pi)^d}\int \mathrm{d}\boldsymbol{k}\{\ln[x+\psi(\boldsymbol{k})] - x/[x+\psi(\boldsymbol{k})]\} - h^2/(Kx). \tag{3}$$

The integral here runs over the Brillouin zone, V_a is the volume of a primitive unit cell, and

$$\psi(\boldsymbol{k}) = [\hat{J}(\mathbf{0}) - \hat{J}(\boldsymbol{k})]/\hat{J}(\mathbf{0}). \tag{4}$$

For totally ferromagnetic interactions ($J(\boldsymbol{l}) \geqslant 0$) we have $0 \leqslant \psi(\boldsymbol{k}) \leqslant 2$. The saddle-point variable x (which is equal to the quantity $\xi_s - 1$ in Joyce's notation) is related to the temperature through the spherical constraint

$$K \equiv \hat{J}(\mathbf{0})/k_B T = R(x) + h^2/Kx^2, \tag{5}$$

$$R(x) = \frac{V_a}{(2\pi)^d}\int \mathrm{d}\boldsymbol{k}[x+\psi(\boldsymbol{k})]^{-1}. \tag{6}$$

On introducing the distribution function

$$W(z) = \frac{V_a}{(2\pi)^d}\int_{\psi(\boldsymbol{k})<z} \mathrm{d}\boldsymbol{k}, \tag{7}$$

we can write (3) and (6) as

$$F_s(x, 0) = \tfrac{1}{2}\int [\ln(x+z) - x/(x+z)]\,\mathrm{d}W(z), \tag{8}$$

$$R(x) = \int (x+z)^{-1}\,\mathrm{d}W(z). \tag{9}$$

2.2. *Expansion for small x*

With a view to elucidating the critical point behaviour ($x \to 0$), we write the derivative of $F_s(x, 0)$ as

$$f(x) \equiv 2\frac{\mathrm{d}F_s(x,0)}{\mathrm{d}x} = (2\pi\mathrm{i})^{-1}\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty}\phi(p)x^{-p}\,\mathrm{d}p, \tag{10}$$

where $\phi(p)$ is given by the Mellin transform (Doetsch 1950)

$$\phi(p) = \int_0^\infty x^{p-1}f(x)\,\mathrm{d}x = \int_0^\infty \mathrm{d}x\int x^p/(x+z)^2\,\mathrm{d}W(z). \tag{11}$$

One must choose $\mathrm{Re}(p)$ and C so that the integral in (11) converges. An analysis of $W(z)$ gives the condition $0 < \mathrm{Re}(p), C < 1$. We interchange the orders of integration in (11) and obtain

$$\phi(p) = \pi p\Omega(p)/\sin(\pi p), \tag{12}$$

where

$$\Omega(p) = \int z^{p-1}\,\mathrm{d}W(z) = (1-p)\int_0^2 W(z)z^{p-2}\,\mathrm{d}z + \sum_i \Delta W(z_i)z_i^{p-1}. \tag{13}$$

Here the $\Delta W(z_i)$ are the magnitudes of possible jumps in $W(z)$ at points z_i. We can now move the path of integration in (10) to the left. At the poles, $-p_k$, of $\phi(p)$ we pick up residues of the integrand and so obtain a expansion of $f(x)$ in terms of powers of x with increasing real parts of the exponents†, namely,

$$f(x) = \sum_k x^{p_k}\,\mathrm{Res}(\phi(-p_k)), \qquad \mathrm{Re}(p_k) \geqslant 0. \tag{14}$$

In order to determine the leading term in this expression we consider the distribution $W(z)$ and hence, through (7), the function $\psi(\boldsymbol{k})$ which represents the interactions. For $\psi(\boldsymbol{k})$ we may assume the leading asymptotic behaviour (Joyce 1966, 1972)

$$\psi(\boldsymbol{k}) = (\Lambda k)^{\bar{\sigma}} + \mathrm{O}(k^{\bar{\sigma}}), \tag{15}$$

with $\bar{\sigma} = 2$ for finite range interactions and $\bar{\sigma} = \min(2, \sigma)$ for inverse power law interactions of the form

$$J(\boldsymbol{l}) \sim |\boldsymbol{l}|^{-d-\sigma}, \qquad \sigma > 0. \tag{16}$$

The length Λ may be identified as the range or scale of the interaction. For $d \leqslant \bar{\sigma}$ the spherical model has no transition so we restrict attention to $d/\bar{\sigma} > 1$. From the definition (7) we then obtain, by simple geometric considerations, the result

$$W(z) = wz^{d/\bar{\sigma}} + \mathrm{O}(z^{d/\bar{\sigma}}) \tag{17}$$

where

$$w = V_d V_a/(2\pi\Lambda)^d \tag{18}$$

and V_d is the volume of a d-dimensional unit sphere.

Inserting (17) in the expression (13) for $\Omega(p)$, we see that as $\mathrm{Re}(p)$ decreases, the integral on the right-hand side of (13) diverges at $\mathrm{Re}(p) = 1-\mathrm{d}/\bar{\sigma}$. Straightforward analytical

† This procedure is used by Barber and Fisher (1973).

continuation then gives

$$\Omega(p) = w2^{p-1+d/\bar{\sigma}}(1-p)/(p-1+d/\bar{\sigma})+(1-p)\int_0^2 (W(z)-wz^{d/\bar{\sigma}})z^{p-2}\,\mathrm{d}z + \sum_i \Delta W(z_i)z_i^{p-1}. \tag{19}$$

The remaining integral is now convergent for $\mathrm{Re}(p) > 1-d/\bar{\sigma}-b$, where $b = \min(2, \sigma/2) - 1$ for $\sigma > 2$ and $b = \min(2, 2/\bar{\sigma})-1$ for $\sigma < 2$.

Under the restriction $1 < d/\bar{\sigma} < 2$, the first singularity of $\phi(p)$ which contributes to (14) appears at $p = 1-d/\bar{\sigma}$, as is obvious from (12) and (19). Evaluation of the residue at this simple pole, using (14) and (18), gives

$$f(x) \approx 2Q_{d,\bar{\sigma}}V_{\mathrm{a}}\Lambda^{-d}x^{d/\bar{\sigma}-1}d/\bar{\sigma}, \qquad (1 < d/\bar{\sigma} < 2) \tag{20}$$

where the universal parameter $Q_{d,\bar{\sigma}}$ is defined by

$$Q_{d,\bar{\sigma}} = \frac{V_d}{2(2\pi)^d}\frac{\pi(d/\bar{\sigma}-1)}{\sin[\pi(d/\bar{\sigma}-1)]}. \tag{21}$$

We may now use (10) which, on integration with respect to x, gives, in leading order in x, the result

$$F_{\mathrm{s}}(x,h) \approx V_{\mathrm{a}}Q_{d,\bar{\sigma}}(x^{1/\bar{\sigma}}/\Lambda)^d\left(1-\frac{h^2\Lambda^d}{R(0)Q_{d,\bar{\sigma}}V_{\mathrm{a}}}x^{-1-d/\bar{\sigma}}\right). \tag{22}$$

A treatment of (9) completely analogous to the treatment of (10) gives

$$R(x) \approx R(0)-2Q_{d,\bar{\sigma}}\frac{d}{d-\bar{\sigma}}V_{\mathrm{a}}\Lambda^{-d}x^{d/\bar{\sigma}-1} \tag{23}$$

to leading order in x.

In the article by Joyce (1972) an alternative way of deriving (23) and hence (22) can be found, which utilizes the properties of Bose–Einstein functions.

2.3. *Scaling form of the quantity Z*

We insert (23) into (5) to obtain

$$\frac{R(0)^2tx^2+h^2}{R(0)x^{d/\bar{\sigma}+1}} \approx 2Q_{d,\bar{\sigma}}\frac{d}{d-\bar{\sigma}}V_{\mathrm{a}}\Lambda^{-d}, \tag{24}$$

where we have utilized the notation

$$t = \frac{R(0)-K}{R(0)} \approx \frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}. \tag{25}$$

The direction of approach to the critical point may be described by the parameter

$$\tau = R(0)^2th^{-2}[x(t,h)]^2. \tag{26}$$

On the critical isotherm we have $\tau = 0$, whereas for $h = 0$, $t \to 0^+$ we have $\tau = \infty$. A one-to-one correspondence between τ and the usual scaling combination of variables $t/|h|^{1/\Delta}$ is given in appendix 1.

Substitution for h^2 in (22) through (24) and the definition (26) gives the result

$$Z(\tau) = Q_{d,\bar{\sigma}}\left(1-\frac{2d}{(d-\bar{\sigma})(1+\tau)}\right)\left(\frac{\xi}{\Lambda x^{-1/\bar{\sigma}}}\right)^d, \tag{27}$$

where we have used (1) with

$$\rho_s = V_a^{-1}. \tag{28}$$

From this result we see that, provided we are able to identify $\Lambda x^{-1/\bar{\sigma}}$ as the correlation length, the combination Z is universal, ie independent of the strength and details of the interaction (other than d and $\bar{\sigma}$) and dependent only on the scaled direction of approach to the critical point. Accordingly we turn to a discussion of the correlation function.

3. Correlation function

The two-point correlation function is given by (Joyce 1972)

$$\Gamma(\boldsymbol{r}, K, h) = \frac{1}{K(2\pi)^d}\int \mathrm{d}\boldsymbol{k}\, \exp\{-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}\}/(x+\psi(\boldsymbol{k})). \tag{29}$$

If we take the lattice Fourier transform and use the expansion (15), this gives,

$$1/\hat{\Gamma}(\boldsymbol{k}, K, h) = V_a K x[1+(\Lambda x^{-1/\bar{\sigma}}k)^{\bar{\sigma}}+\mathrm{O}(k^{\bar{\sigma}})]. \tag{30}$$

In the case of finite-range interactions the quantity $\Lambda x^{-1/2}$ is by definition (Fisher 1964, Fisher and Burford 1967) equal to the effective correlation length, ξ_1, determined by the second moment of the correlation function (see appendix 2, (A2.9)). In leading order in x, ie on approaching the critical point, ξ_1 becomes equal to the true correlation range, ξ_0, (Joyce 1972), which determines the exponential decay of $\Gamma(\boldsymbol{r}, K, h)$ (Fisher 1964).

For inverse power law interactions of the form (16) it is clear that the true correlation length ξ_0 is an inappropriate concept. However, for $\sigma < 2$ the second moment correlation length ξ_1 is also divergent. Nevertheless, the correlation lengths ξ_p, defined via moments of order $0 < 2p < \sigma$, do exist. In the appendix 2 we show that in this case an appropriately defined $\bar{\sigma}$th moment correlation length, $\tilde{\xi}$, is, up to a universal constant depending only on d and $\bar{\sigma}$, indeed equal to $\Lambda x^{-1/\bar{\sigma}}$. This completes our derivation.

4. Discussion

In order to compare our results with the data of Stauffer *et al* (1972) we calculate the quantity

$$X_\infty \equiv (2-\alpha)(1-\alpha)(-\alpha)Z(\infty), \tag{31}$$

which corresponds to their parameter X. We specialize to the case of short-range interactions and three dimensions. Using a generalized definition of the specific heat exponent α (Fisher 1967) we have (Joyce 1972)

$$\alpha = (d-2\bar{\sigma})/(d-\bar{\sigma}) \tag{32}$$

which in our case gives $\alpha = -1$. Thus we find

$$X_\infty = (4\pi)^{-1} = 0{\cdot}079577\ldots. \tag{33}$$

This value may be compared with

$$\begin{aligned} X_1 &= (2\pi)^{-1} = 0{\cdot}1591549\ldots, && \text{(Ising, } d = 2\text{)},\\ X_1 &= 0{\cdot}0165 \pm 0{\cdot}0001, && \text{(Ising, } d = 3\text{)},\\ X_3 &= 0{\cdot}076 \pm 0{\cdot}002, && \text{(Heisenberg, } d = 3\text{)}. \end{aligned} \tag{34}$$

These values are taken from the paper of Stauffer *et al* (1972) where detailed references are given. One would expect X_3 for the $d = 3$ Heisenberg model to lie between the values X_1 for the Ising model ($n = 1$) and X_∞ for the spherical model ($n \to \infty$). The uncertainty in the numerical result allows only for a surprisingly small difference between the Heisenberg and spherical models.

On the critical isotherm we may calculate the quantity

$$Y \equiv \chi h^2 \xi^d = -Z(0)(\delta+1)/\delta^2 \tag{35}$$

where χ is the reduced (dimensionless) isothermal susceptibility per unit volume. For three dimensions and finite-range interactions one obtains, with $\delta = 5$ (for a definition of the critical exponent δ see Fisher 1967)

$$Y_\infty = (20\pi)^{-1} = 0{\cdot}015915\ldots. \tag{36}$$

This may be compared with values for the Ising model in two and three dimensions, which follow from the second moment correlation length and the susceptibility on the critical isotherm, calculated by Tarko and Fisher (1973a, b, Tarko 1974),

$$\begin{aligned} Y_1 &= 0{\cdot}00383 \pm 0{\cdot}00013, && (d = 2, \text{square}),\\ Y_1 &= 0{\cdot}0044 \pm 0{\cdot}0004, && (d = 3, \textsc{sc}),\\ Y_1 &= 0{\cdot}0046 \pm 0{\cdot}0006, && (d = 3, \textsc{bcc}). \end{aligned} \tag{37}$$

The agreement between the SC and BCC values represents an explicit test of the Stauffer *et al* hypothesis as extended to the magnetic field dependence.

In (27) Z has been calculated for a Bravais lattice. However, the results for the Ising models, mentioned in the introduction, suggest that (27) is most probably also independent of this aspect of the lattice structure.

Finally we conclude that for the spherical model the hypothesis of Stauffer *et al* (1972) holds true as a special case of the universality of $Z(\tau)$ for $1 < d/\bar{\sigma} < 2$. However, we should also observe that for $d/\bar{\sigma} > 2$ there is still a universal quantity $Z(\infty)$. But in this case the singular terms are dominated by analytic ones which in fact, determine the leading asymptotic critical behaviour and, as well known, yield mean-field exponents. For integral $d/\bar{\sigma}$ there are also logarithmic corrections in the parameter x, which we have not discussed explicitly. Finally, we see from (32), that 'accidental' analyticity of the leading singular part of F_s occurs for integral values of $d/(d-\bar{\sigma})$ when $d/\bar{\sigma} < 2$.

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Appendix 1. Identification of a scaling variable

Elimination of the variable x in (24) through (26) gives

$$t/|h|^{2(d-\bar{\sigma})/(d+\bar{\sigma})} = D\tau/(1+\tau)^{2\bar{\sigma}/(d+\bar{\sigma})} \tag{A1.1}$$

with

$$D = (2Q_{d,\bar{\sigma}}R(0)\frac{d}{d-\bar{\sigma}}V_{\mathrm{a}}\Lambda^{-d})^{2\bar{\sigma}/(d+\bar{\sigma})}R(0)^{-2}. \tag{A1.2}$$

The left-hand side of (A1.1) corresponds to the usual scaling combination of variables $t/|h|^{1/\Delta}$ with

$$\Delta = \tfrac{1}{2}+\bar{\sigma}/(d-\bar{\sigma}). \tag{A1.3}$$

Since $2\bar{\sigma}/(d+\bar{\sigma}) < 1$, equation (A1.1) provides a one-to-one correspondence between the variables $t/|h|^{1/\Delta}$ in the interval $(-\infty, \infty)$ and τ in the interval $(-1, \infty)$. From (27), (1) and (24), (26) one obtains for the free energy the expression

$$F_{\mathrm{s}} \approx F_0 t^{2-\alpha}\left(\frac{1+\tau}{\tau}\right)^{2-\alpha}\left(1-\frac{2d}{(d-\bar{\sigma})(1+\tau)}\right) \tag{A1.4}$$

where the specific heat exponent α is given by (32) and

$$F_0 = \left(\frac{d-\bar{\sigma}}{2d}\frac{R(0)\Lambda^{\bar{\sigma}}}{(V_{\mathrm{a}}Q_{d,\bar{\sigma}})^{\bar{\sigma}/d}}\right)^{d/(d-\bar{\sigma})}. \tag{A1.5}$$

Since (A1.1) provides a universal relation between τ and $D^{-1}t/|h|^{1/\Delta}$ the result (A1.4) can be written as

$$F_{\mathrm{s}} \approx F_0 t^{2-\alpha}\phi(D^{-1}t/|h|^{1/\Delta}) \tag{A1.6}$$

where $\phi(x)$ is a universal function determined by d and $\bar{\sigma}$, with $\phi(\infty) = 1$. This is one of the standard scaling forms for the free energy.

The parameter τ corresponds to the quantity $Y_{\pm}$, which is discussed in the paper by Joyce (1972). The connection is

$$\tau = \pm[Y_{\pm}(D^{\Delta}h/|t|^{\Delta})]^{-2}. \tag{A1.7}$$

Appendix 2. Correlation moments for long-range interactions

Following Theuman (1970) we calculate an angular average of $\Gamma(\boldsymbol{r}, K, h)$ in (29) and take a continuous approximation for the lattice. This is justified near the critical point and yields

$$\Gamma(r, K, h) \approx C\Lambda^{\bar{\sigma}}\int\frac{\mathrm{d}\boldsymbol{k}}{x+\psi(\boldsymbol{k})}(kr)^{1-d/2}\mathrm{J}_{d/2-1}(kr) \tag{A2.1}$$

where $J_{d/2-1}(z)$ is a Bessel function of first kind and $C\Lambda_{\bar{\sigma}}$ is proportional to T and depends on the lattice. As before Λ is defined in (15). We use the truncated expansion (15) and the usual definition $\kappa = x^{1/\bar{\sigma}}/\Lambda$ and change variables to $y = k/\kappa$ which then gives

$$\Gamma(r, K, h) = C\kappa^{d-\bar{\sigma}} \int \frac{\mathrm{d}y}{1+y^{\bar{\sigma}}} (\kappa r y)^{1-d/2} \mathscr{J}_{d/2-1}(\kappa r y). \tag{A2.2}$$

The integral is now over a volume similar to the Brillouin zone with all lengths scaled by the factor κ^{-1}. For the Bessel function we may utilize the integral representation (see eg Abramowitz *et al* 1970)

$$\mathrm{J}_\nu(z) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}t \frac{\Gamma(-t)}{\Gamma(\nu+t+1)} (z/2)^{\nu+2t} \qquad (c < 0, z > 0). \tag{A2.3}$$

Interchange of the orders of integration subject to the condition $2c > -d$ then leads to

$$\Gamma(r, K, h) = C\kappa^{d-\bar{\sigma}} \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}t \frac{\Gamma(-t)}{\Gamma(d/2+t)} (\kappa r/2)^{2t} \int dy y^{2t}/(1+y^{\bar{\sigma}}). \tag{A2.4}$$

Since we are interested in the limit $\kappa \to 0$, we can approximate the y integral by extending the range of integration over the whole y space, provided we impose the additional constraint $2c < -d+\bar{\sigma}$. After performing this integral we are left with

$$\Gamma(r, K, h) = C_1\kappa^{d-\bar{\sigma}} \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}t (\kappa r/2)^{2t} \Gamma(-t) \left[\Gamma(d/2+t) \sin\left(\pi \frac{2t+d}{\bar{\sigma}}\right) \right]^{-1}. \tag{A2.5}$$

The path of integration can now be chosen to satisfy $-d-\bar{\sigma} < 2c < -d+\bar{\sigma}$, since the singularity of the reciprocal sine at $2t = -d$ cancels against the zero of $\Gamma^{-1}(d/2+t)$. We now take the qth moment

$$M_q(K, h) = \int_a^\infty r^{d+q-1} \Gamma(r, K, h) \, \mathrm{d}r, \qquad q < \bar{\sigma} \tag{A2.6}$$

and notice that the orders of t and r integrations can be interchanged when we choose $-d-\bar{\sigma} < 2c < -d-q$. Performing the r integral yields

$$\begin{aligned} M_q(K, h) = {} & -\frac{C_1}{2\pi\mathrm{i}} \kappa^{-\bar{\sigma}-q} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}t 2^{-2t} (a\kappa)^{2t+d+q} \Gamma(-t) \\ & \times \left[\Gamma(d/2+t) \sin\left(\pi \frac{2t+d}{\bar{\sigma}}\right) (2t+d+q) \right]^{-1}. \end{aligned} \tag{A2.7}$$

Finally we move the path of integration to the right and pick up the residues of the singularities that are passed. This yields a series of increasing powers of κa. Since we are interested in the limit $\kappa \to 0$ we retain only the first term, in which the a dependence drops out and yields

$$M_q(K, h) \approx C_1 2^{d+q-1} \kappa^{-\bar{\sigma}-q} \Gamma\left(\frac{d+q}{2}\right) \left[\Gamma\left(-\frac{q}{2}\right) \sin\left(-\frac{\pi q}{\bar{\sigma}}\right) \right]^{-1}. \tag{A2.8}$$

As explained in the text, the second moment M_2 diverges for $\sigma < 2$ and so we may introduce the generalized q correlation length, $\xi_{q/2}$, defined by (Fisher 1969)

$$(\xi_{q/2})^q = 2^{-q} \Gamma\left(\frac{d}{2}\right) M_q(K, h) \left[\Gamma\left(\frac{d+q}{2}\right) M_0(K, h) \right]^{-1}. \tag{A2.9}$$

For short-range interactions, $\bar{\sigma} = 2$, this goes into the usual effective correlation length defined through the second moment: $[\xi_1]^2 = M_2(K, h)/(2dM_0(K, h))$. From (A2.8) we then have

$$[\xi_{q/2}]^q = \kappa^{-q} 2\pi/[\bar{\sigma} \sin(-\pi q/\bar{\sigma})\Gamma(-q/2)]. \tag{A2.10}$$

In order to avoid the arbitrary q dependence we may define a limiting correlation length, $\tilde{\xi}$, by the relation

$$\tilde{\xi}^{\sigma} = \lim_{q \to \bar{\sigma}^-} \frac{\bar{\sigma} - q}{2 - q} (\xi_{q/2})^q. \tag{A2.11}$$

The result (A2.10) and this definition then yield the simple result

$$\tilde{\xi} \approx \Lambda x^{-1/\bar{\sigma}}/\tilde{f} \tag{A2.12}$$

with

$$\tilde{f}^{\sigma} = (\bar{\sigma}/2 - 1)\Gamma(-\bar{\sigma}/2). \tag{A2.13}$$

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